GENERAL BOUNDARY VALUE PROBLEMS FOR LINEAR DIFFERENTIAL EQUATIONS

By

KAJ LEO NIELSEN

A. B., University of Michigan, 1936

A. M., Syracuse University, 1937

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN
MATHEMATICS IN THE GRADUATE SCHOOL OF
THE UNIVERSITY OF ILLINOIS, 1940

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Introduction

Let $L(x, \lambda; y)$ be the differential polynomial

$$(A) \qquad L(x, \lambda; y) = \sum_{k=0}^{n} \lambda^{H(n-k)} a_{n-k}(x, \lambda) y^{(k)}(x, \lambda)$$

$$[y^{(k)}(x, \lambda), (k \geq 0), \text{ denotes } \frac{\partial^k y}{\partial x^k}]$$

where the coefficients, $a_{n-k}(x, \lambda)$, are assumed to be indefinitely differentiable in x, for x on a closed real interval (a,b); $[a \leq x \leq b]$, and analytic in λ for $|\lambda| \geq R_0 > 0$, $(\lambda \neq \infty)$, and being asymptotic to a series of the form

$$a_{n-k}(x, \lambda) \sim \sum_{\nu=0}^{\infty} \beta_{n-k}(x) \lambda^{-\nu} \qquad [k = 0,1,\ldots,n; \quad x \text{ in } (a,b)].$$

We shall use the term "asymptotic" in the ordinary sense as defined in (T)[1].

One formulation of the boundary value problem is to determine a function $y(x, \lambda)$ which satisfies

$$(1) \qquad L(x, \lambda; y) = f(x)$$

and the n boundary conditions

$$(1a) \qquad M_i(y) = \sum_{k=1}^{n} \int_c^d y^{(k-1)}(t, \lambda) d\alpha_{i,k}(t) = 0 \qquad [i = 1,\ldots,n; \; a \leq c < d \leq b]$$

where

(i) $\alpha_{i,k}(t)$ is of bounded variation and the integrals are in the sense of Stieltjes, and

(ii) the $M_i(y)$ are linearly independent.

When $f(x) = 0$ the problem is called homogeneous; otherwise it is called non-homogeneous. The homogeneous problem is said to be compatible if it admits a solution which is not identically zero; if it does not admit a solution which is not identically zero it is said to be incompatible.

Associated with the homogeneous equation (1); i.e. $f(x) = 0$, is the system which in matrix notation will be written as

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[1] W.J. Trjitzinsky: "Theory of Linear Differential Equations Containing a Parameter"; Acta Mathematica, vol. 67 (1936); p. 2; in the sequel referred to as (T).

(B) $$Y^{(1)}(x, \lambda) = Y(x, \lambda)D(x, \lambda)$$

where

$$Y(x, \lambda) = (y_{i,j}(x,\lambda)) = (y_i^{(j-1)}(x,\lambda)); \qquad Y^{(1)}(x, \lambda) = (y_{i,j}^{(1)}(x,\lambda))$$

and

$$D(x, \lambda) = \begin{Vmatrix} 0 & 0 & \dots & . & \dots & 0 & -\lambda^{Hn} a_n(x,\lambda) \\ 1 & 0 & \dots & . & \dots & 0 & -\lambda^{H(n-1)} a_{n-1}(x,\lambda) \\ . & . & \dots & . & \dots & . & . \\ 0 & 0 & \dots & . & \dots & 1 & -\lambda^{H} a_1(x,\lambda) \end{Vmatrix}$$

If we consider this equation with the boundary condition

(2) $$Y(c, \lambda)W_c + Y(d, \lambda)W_d = 0$$

where $W_c = (w_{i,j}^{[c]})$, $W_d = (w_{i,j}^{[d]})$ are matrices of constant terms so that $|W_c| \neq 0$, $|W_d| \neq 0$, we have a homogeneous boundary value problem.

The associated non-homogeneous problem is formulated by the system

(C) $$Y^{(1)}(x, \lambda) = Y(x, \lambda)D(x, \lambda) + B(x)$$

and the equation (2).

Trjitzinsky[1] has found the solution of (B) to be the matrix

(3) $$Z(x, \lambda) = {}_rY^{-1}(c, \lambda)\,{}_rY(x, \lambda)$$

for x in (c,d) and for λ in a region R, where R is a region in the complex λ-plane bounded by regular curves and extending to ∞ and nowhere intersecting for $|\lambda| \geq \rho_o > 0$, where ρ_o is sufficiently great. By taking the interval (c,d) sufficiently small the neighborhood of $\lambda = \infty$ is divided into a finite number of these regions, R_ν, $[\nu = 1,\dots,N]$.

Substituting this solution into the boundary condition (2) we get

$${}_rY^{-1}(c, \lambda)\,{}_rY(c, \lambda)W_c + {}_rY^{-1}(c, \lambda)\,{}_rY(d, \lambda)W_d = 0$$

or

(4) $$W_c + {}_rY^{-1}(c, \lambda)\,{}_rY(d, \lambda)W_d = 0.$$

The determinant of this expression is

(5) $$\triangle(\lambda) = |W_c + {}_rY^{-1}(c, \lambda)\,{}_rY(d, \lambda)W_d|$$

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[1] See (T), the Fundamental Existence Theorem, p. 36.

and the values for which $\triangle(\lambda) = 0$ are called characteristic values and the solutions of the problem (B),(2) which exist at these values for λ are called characteristic solutions. We shall call $\triangle(\lambda)$ the characteristic function.

A necessary and sufficient condition that the non-homogeneous system has a unique solution is that the corresponding homogeneous system is incompatible; i.e. $\triangle(\lambda) \neq 0$. In this paper we are concerned with the problem of determining the values of λ for which the characteristic function, $\triangle(\lambda)$, $\neq 0$. In Part I we discuss this problem for the system (C),(2) and extend the problem for the n-point boundary value problem and the ∞ - point boundary value problem. In Part II we discuss this problem for the equation (1) and the conditions (1a).

PART I

Determination of the λ-Region for the Problems Connected with (C).

1. The Two-Point Problem; (C),(2).

"For what values of λ is the non-homogeneous problem (C),(2) possible when (c,d) is the interval for which the Trjitzinsky Fundamental Theorem holds."

By (T) [p.33] we have the asymptotic expression

$$(6) \qquad Z(x,\lambda) \sim \left(\sum_{s=1}^{n} e^{Q_s(x,\lambda)-Q_s(c,\lambda)} \sum_{\nu=0}^{\infty} z_{i,j,s:\nu}(x)\, \lambda^{-\nu/k} \right)$$

for x in (c,d); λ in R, so that we have

$$(7) \qquad \triangle(\lambda) \sim \left| W_c + \left(\sum_{s=1}^{n} e^{Q_s(d,\lambda) - Q_s(c,\lambda)} \sum_{\nu=0}^{\infty} z_{i,j,s:\nu}(c,d)\, \lambda^{-\nu/k} \right) W_d \right|$$

or

$$(8) \quad \triangle(\lambda) \sim \left| \left(w_{i,j}^{[c]} + \sum_{s=1}^{n} e^{Q_s(d,\lambda) - Q_s(c,\lambda)} \sum_{r=1}^{n} \sum_{\nu=0}^{\infty} z_{i,r,s:\nu}(c,d) w_{r,j}^{[d]}\, \lambda^{-\nu/k} \right) \right|.$$

We shall now introduce the following definitions:

Definition 1.

Let the totality (if any) of the identically vanishing functions $Q_s(d,\lambda) - Q_s(c,\lambda)$ be denoted by

$$(9) \qquad P_{s_w} = Q_{s_w}(d,\lambda) - Q_{s_w}(c,\lambda) \qquad [w = 1,\ldots,p].$$

Definition 2.

Any particular region R_1 of the R_ν may contain a finite set of non-overlapping closed subregions, $R^1_{\nu,i}$ $[i = 1,\ldots,N_\nu]$, each extending to infinity and such that

$$(10) \qquad \lim \left| e^{Q_s(d,\lambda)} - Q_s(c,\lambda) \right| = 0$$

for every $s \neq s_w$ $[w = 1,\ldots,p]$; the limit being taken when $\lambda \to \infty$ in a region $R^1_{\nu,i}$ $[i = 1,\ldots,N_\nu]$. We shall define R' as a particular set of regions $R^1_{\nu,i}$ $[i = 1,\ldots,N_\nu;\ \nu = 1,\ldots,N]$.

Definition 3.

Let the matrix

$$(11) \qquad ({}_0g_{i,j}) = \left(w^{[c]}_{i,j} + \sum_{w=1}^{p} \sum_{r=1}^{n} z_{i,r,s_w:0}(c,d) w^{[d]}_{r,j} \right).$$

Consider now

$$(12) \qquad \sum_{s=1}^{n} e^{Q_s(d,\lambda)} - Q_s(c,\lambda) \sum_{r=1}^{n} \sum_{\nu=0}^{\infty} z_{i,r,s:\nu}(c,d) w^{[d]}_{r,j} \lambda^{-\nu/k}$$

$$= \sum_{w=1}^{p} e^{P_{s_w}} \sum_{r=1}^{n} z_{i,r,s_w:0}(c,d) w^{[d]}_{r,j} + \sum_{s \neq s_w}^{n} e^{P_s} \sum_{r=1}^{n} z_{i,r,s:0}(c,d) w^{[d]}_{r,j}$$

$$+ \sum_{w=1}^{p} e^{P_{s_w}} \sum_{r=1}^{n} z_{i,r,s_w:1}(c,d) w^{[d]}_{r,j} \lambda^{-1/k} + \sum_{s \neq s_w}^{n} e^{P_s} \sum_{r=1}^{n} z_{i,r,s:1}(c,d) w^{[d]}_{r,j} \lambda^{-1/k}$$

$$+ \ldots\ldots\ldots\ldots \qquad \text{where } P_s \text{ denotes } Q_s(d,\lambda) - Q_s(c,\lambda).$$

Now using Definition 2 above we see that

$$(13) \qquad \sum_{s \neq s_w}^{n} e^{P_s} \sum_{r=1}^{n} \sum_{\nu=0}^{\infty} z_{i,r,s:\nu}(c,d) w^{[d]}_{r,j} \lambda^{-\nu/k} = \phi_{i,j} \sim 0 \qquad [\text{for } \lambda \text{ in } R'].$$

Further using Definitions 1 and 3 and letting

$$(14) \qquad \sum_{w=1}^{p} \sum_{r=1}^{n} z_{i,r,s_w:\nu}(c,d) w^{[d]}_{r,j} = {}_\nu g_{i,j} \qquad [\nu = 1,2,\ldots.]$$

since they are constants; we thus have

$$(15)\qquad \triangle(\lambda) \sim \left| ({}_{o}g_{i,j}) + \left(\sum_{\nu=1}^{\infty} {}_{\nu}g_{i,j}\lambda^{-\nu/k} \right) + \phi_{i,j} \right|$$

$$\sim \left| \left({}_{o}g_{i,j} + \sum_{\nu=1}^{\infty} {}_{\nu}g_{i,j}\lambda^{-\nu/k} \right) \right|$$

$$\sim G_o + G_1\lambda^{-1/k} + \ldots + G_\nu\lambda^{-\nu/k} + \ldots\ldots\ldots$$

where the G_ν's are combinations of the constants ${}_{\nu}g_{i,j}$ $[\nu = 0,1,2,\ldots]$. Therefore we have

$$(16)\qquad \triangle(\lambda) = G_o + G_1\lambda^{-1/k} + \ldots + G_{m-1}\lambda^{-(m-1)/k} + G_m(\lambda)\lambda^{-m/k}$$

$[m = 1,2,\ldots$ and where $|G_m(\lambda)| \leq M_m$; λ in $R']$.

We shall now consider the following cases:

Case I:

Suppose $G_o = G_1 = \ldots = G_{\nu-1} = 0$; $G_\nu \neq 0$, $[\nu < m]$ and (16) becomes

$$(17)\qquad \triangle(\lambda) = G_\nu\lambda^{-\nu/k} + G_{\nu+1}\lambda^{-(\nu+1)/k} + \ldots + G_m(\lambda)\lambda^{-m/k}, \quad [\nu < m];$$

multiplying through by $\lambda^{\nu/k}$ we obtain

$$(18)\qquad \lambda^{\nu/k}\triangle(\lambda) = G_\nu + \xi(\lambda); \text{ where } \lim_{\lambda\to\infty} \xi(\lambda) = 0.$$

Thus for $|\lambda| \geq \rho_1$; λ in R' and since $G_\nu \neq 0$, we have

$$(19)\qquad \lambda^{\nu/k}\triangle(\lambda) \neq 0 \text{ and therefore } \triangle(\lambda) \neq 0.$$

Case II:

Suppose $G_o = G_1 = \ldots = G_{m-1} = 0$; then (16) becomes $\triangle(\lambda) = G_m(\lambda)\lambda^{-m/k}$ where $|G_m(\lambda)| \leq M_m$; λ in R' and $\triangle(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow \infty$ and nothing more can be said about the characteristic function in this case.

Hence the following Theorem can be stated:

THEOREM I.

If in the non-homogeneous boundary value problem (C),(2) we define R' as in Definition 2, the problem will then be possible for every λ in R', providing $|\lambda| \geq \rho_1$ and some $G_\nu \neq 0$, $[\nu < m]$. Here ρ_1 is a fixed number depending upon the choice of R'.

If we now take $m = \nu + 1$; (16) becomes

$$(20)\quad \lambda^{\nu/k}\triangle(\lambda) = G_\nu + G_{\nu+1}(\lambda)\lambda^{-1/k} \quad \text{where } |G_{\nu+1}(\lambda)\lambda^{-1/k}| \le M_{\nu+1}\rho_1^{-1/k}$$

[for λ in R' with $|\lambda| \ge \rho_1$].

Take $\rho \ge \rho_1$ and so that

$$(21)\quad M_{\nu+1}\rho^{-1/k} \le \tfrac{1}{2}|G_\nu|; \text{ then } |\lambda^{\nu/k}\triangle(\lambda)| \ge \tfrac{1}{2}|G_\nu|$$

for λ in R', $|\lambda| \ge \rho$; and we have

$$(22)\quad \tfrac{1}{2}|G_\nu| \le |\lambda^{\nu/k}\triangle(\lambda)| \le |G_\nu| + \tfrac{1}{2}|G_\nu| = \tfrac{3}{2}|G_\nu|; \text{ for } |\lambda| \ge \rho.$$

Hence the following Corollary can be stated:

Corollary 1.

By further choosing ρ to satisfy condition (21) and letting $m = \nu + 1$ we have $\lambda^{\nu/k}\triangle(\lambda)$ restricted between $\frac{1}{2}|G_\nu|$ and $\frac{3}{2}|G_\nu|$.

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2. The N-Point Problem; (C),(23).

If we choose any finite set of points $a_1, a_2, \ldots, a_n$ in our x interval (c,d) our boundary condition could then be written:

$$(23)\quad \sum_{\mu=1}^{n} Y(a_\mu, \lambda)W_{a_\mu} = 0$$

where W_{a_μ} are constant matrices such that $|W_{a_\mu}| \neq 0$; [$\mu = 1,\ldots,n$].

If we substitute the solution $Z(x,\lambda) = {}_rY^{-1}(c,\lambda)\,{}_rY(x,\lambda)$ into this boundary condition we get

$$(24)\quad \sum_{\mu=1}^{n} {}_rY^{-1}(c,\lambda)\,{}_rY(a_\mu,\lambda)W_{a_\mu} = 0.$$

The determinant of the left hand side of this equation is our characteristic function, $\triangle(\lambda)$. Now using the asymptotic expression (6) we have

$$(25)\quad \triangle(\lambda) \sim \left| \left(\sum_{s=1}^{n} e^{Q_s(a_1,\lambda) - Q_s(c,\lambda)} \sum_{\nu=0}^{\infty} z_{i,j,s:\nu}(c,a_1)\lambda^{-\nu/k} \right) W_{a_1} + \right.$$

$$\left. \cdots + \left(\sum_{s=1}^{n} e^{Q_s(a_n,\lambda) - Q_s(c,\lambda)} \sum_{\nu=0}^{\infty} z_{i,j,s:\nu}(c,a_n)\lambda^{-\nu/k} \right) W_{a_n} \right| .$$

Let us now consider the general term of the sum that constitutes the elements of this determinant

$$(26)\qquad \sum_{s=1}^{n} e^{Q_s(a_t,\lambda) - Q_s(c,\lambda)} \sum_{r=1}^{n}\sum_{\nu=0}^{\infty} z_{i,r,s:\nu}(c,a_t) w_{r,j}^{[t]} \lambda^{-\nu/k}$$

where $w_{i,j}^{[t]}$ are elements of W_{a_t} [t = 1,...,n]. If we now let $P_s^t = Q_s(a_t,\lambda) - Q_s(c,\lambda)$ and $P_{s_w}^t = Q_{s_w}(a_t,\lambda) - Q_{s_w}(c,\lambda)$ we can break (26) up into the part with e to the exponent P_s^t and the part with e to the exponent $P_{s_w}^t$. Now using Definition 2 with d replaced by a_t we see that

$$(27)\qquad \sum_{s\neq s_w}^{n} e^{P_s^t} \sum_{r=1}^{n}\sum_{\nu=0}^{\infty} z_{i,r,s:\nu}(c,a_t) w_{r,j}^{[t]} \lambda^{-\nu/k} = \phi_{i,j}^{[t]} \sim 0$$

[for λ in R'; t = 1,...,n].

Further using Definition 1, $P_{s_w}^t = 0$, we see that

$$(28)\qquad \triangle(\lambda) \sim \left|\left(\sum_{w=1}^{p}\sum_{r=1}^{n}\sum_{\nu=0}^{\infty} z_{i,r,s_w:\nu}(c,a_1) w_{r,j}^{[1]} \lambda^{-\nu/k} + \cdots\cdots + \sum_{w=1}^{p}\sum_{r=1}^{n}\sum_{\nu=0}^{\infty} z_{i,r,s_w:\nu}(c,a_n) w_{r,j}^{[n]} \lambda^{-\nu/k}\right)\right|.$$

If we now write out this expression and collect the coefficients of the various powers of λ, we shall have (28) in the form

$$\left|\left(\left\{\sum_{w=1}^{p}\sum_{r=1}^{n} z_{i,r,s_w:0}(c,a_1) w_{r,j}^{[1]} + \sum_{w=1}^{p}\sum_{r=1}^{n} z_{i,r,s_w:0}(c,a_2) w_{r,j}^{[2]} + \cdots + \sum_{w=1}^{p}\sum_{r=1}^{n} z_{i,r,s_w:0}(c,a_n) w_{r,j}^{[n]}\right\}\right.$$

$$+ \left\{\sum_{w=1}^{p}\sum_{r=1}^{n} z_{i,r,s_w:1}(c,a_1) w_{r,j}^{[1]} + \sum_{w=1}^{p}\sum_{r=1}^{n} z_{i,r,s_w:1}(c,a_2) w_{r,j}^{[2]} + \cdots + \sum_{w=1}^{p}\sum_{r=1}^{n} z_{i,r,s_w:1}(c,a_n) w_{r,j}^{[n]}\right\} \lambda^{-1/k}$$

$$\left. + \{\cdots\cdots\} \lambda^{-2/k} + \{\cdots\cdots\} \lambda^{-3/k} + \cdots\cdots\cdots\cdots\cdots\right)\Big|.$$

Now let

$$(29)\qquad {}_{\nu}g_{i,j} = \sum_{w=1}^{p}\sum_{r=1}^{n} z_{i,r,s_w:\nu}(c,a_1)w_{r,j}^{[1]} + \cdots\cdots + \sum_{w=1}^{p}\sum_{r=1}^{n} z_{i,r,s_w:\nu}(c,a_n)w_{r,j}^{[n]} \qquad [\nu = 0,1,2,\ldots]$$

and we have

$$(30)\qquad \triangle(\lambda) \sim \left| {}_{0}g_{i,j} + {}_{1}g_{i,j}\lambda^{-1/k} + {}_{2}g_{i,j}\lambda^{-2/k} + \cdots\cdots \right| \sim G_0 + G_1\lambda^{-1/k} + G_2\lambda^{-2/k} + \ldots + G_{\nu}\lambda^{-\nu/k} + \cdots\cdots$$

where the G_{ν}'s are constants obtained by combinations of the ${}_{\nu}g_{i,j}$.
Hence finally

$$(31)\qquad \triangle(\lambda) = G_0 + G_1\lambda^{-1/k} + \ldots + G_{m-1}\lambda^{-(m-1)/k} + G_m(\lambda)\lambda^{-m/k}$$

where $|G_m(\lambda)| \leq M_m$ for λ in R'.

We can now consider the two cases as before and thus the following Theorem can be stated:

THEOREM II.

If we choose our boundary condition to be at a finite set of points, $a_1, a_2, \ldots, a_n$, in our x interval (c,d); i.e. condition (23); then the non-homogeneous boundary value problem (C),(23) will be possible for every λ in R' providing $|\lambda| \geq \rho_1$ and some $G_{\nu} \neq 0$, $[\nu < m]$. The G_{ν} now depending upon (29) for its value.

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3. The ∞ - Point Problem; (C),(32).

If we further choose a denumerably infinite number of points, $a_1, a_2, \ldots, a_n, \ldots$, in our x interval (c,d) we could formulate a boundary value problem with the equation (c) and the condition

$$(32)\qquad \sum_{\mu=1}^{\infty} Y(a_{\mu},\lambda)W_{a_{\mu}} = 0$$

where $W_{a_{\mu}}$ are constant matrices such that $|W_{a_{\mu}}| \neq 0$.

With this condition our characteristic function would have the form

$$(33)\qquad \triangle(\lambda) = \left| {}_{r}Y^{-1}(c,\lambda)\,{}_{r}Y(a_1,\lambda)W_{a_1} + \cdots\cdots + {}_{r}Y^{-1}(c,\lambda)\,{}_{r}Y(a_n,\lambda)W_{a_n} + \cdots\cdots\cdots \right|.$$

Our asymptotic relations would hold as before and we could obtain the relationship (30) except that our coefficients of the various powers of λ are in this case an infinite series; i.e.

$$(34) \qquad {}_{\nu}g_{i,j} = \sum_{w=1}^{p}\sum_{r=1}^{n} z_{i,r,s_w:\nu}(c,a_1)w_{r,j}^{[1]} + \cdots + \sum_{w=1}^{p}\sum_{r=1}^{n} z_{i,r,s_w:\nu}(c,a_n)w_{r,j}^{[n]} + \cdots \qquad [\nu = 0,1,2,\ldots]$$

and before we can obtain a relationship like (31) it becomes necessary that this sum be convergent. Since the $z_{i,r,s_w:\nu}(c,a_\mu)$ are constants, obtained by substituting the points a_μ into the solution of the particular differential equation under consideration and the $w_{r,j}^{[\mu]}$ are the elements of the constant matrix W_{a_μ}; it becomes necessary to place the restrictions on the latter. We shall now assume that it is possible to choose the elements $w_{r,j}^{[\mu]}$ so that (34) converges, say to ${}_{\nu}g'_{i,j}$ $[\nu = 0,1,2,\ldots]$.

Under these conditions we shall again have equation (31) with the G_ν's depending upon ${}_{\nu}g'_{i,j}$ for their values and the following Theorem can be stated:

THEOREM III.

If we choose our boundary condition to be at a denumerably infinite number of points; i.e. condition (32); and further if we can choose the elements $w_{i,j}^{[\mu]}$ so that (34) converges then the non-homogeneous problem (C),(32) will be possible for every λ in R' providing $|\lambda| \geq \rho_1$ and some $G_\nu \neq 0$, $[\nu < m]$; the G_ν depending upon ${}_{\nu}g'_{i,j}$ for its value.

PART II

Determination of the λ-Region for the Problem (1),(1a).

We shall now consider the problem:

"For what values of λ is the non-homogeneous problem (1),(1a) possible when L is the unrestricted operator (A) and (c,d), (c < d), is the interval for which the Trjitzinsky Fundamental Existence Theorem holds."

In the equations

$$(1a) \qquad M_i(y) = \sum_{k=1}^{n} {}_c\int^d y^{(k-1)}(t,\lambda)\,d\alpha_{i,k}(t) = 0 \qquad [i = 1,\ldots,n]$$

where $\alpha_{i,k}(t)$ is a function of bounded variation and the integrals are in the sense of Stieljtes, we have that $\alpha_{i,k}(t) = \sigma_{i,k}(t) + \tau_{i,k}(t)$, where $\sigma_{i,k}(t)$ is a continuous function and $\tau_{i,k}(t)$ is a step function.

Denoting by $a_1, a_2, \ldots, a_n, \ldots$ the points of discontinuity of the function $\tau_{i,k}(t)$, the end points of (c,d) inclusive if need be, we can then write

$$(35) \quad M_i(y) = \sum_{k=1}^{n} {}_c\int^d y^{(k-1)}(t,\lambda)\, d\,\sigma_{i,k}(t) + \sum_{k=1}^{n}\sum_{\mu=1}^{\infty} y^{(k-1)}(a_\mu,\lambda) w_{i,k}^{[\mu]}$$

where $w_{i,k}^{[\mu]}$ are constants determined by the jumps of the function $\tau_{i,k}(t)$.

Let

$$N_1(y) = \sum_{k=1}^{n} {}_c\int^d y^{(k-1)}(t,\lambda)\, d\,\sigma_{i,k}(t) \quad \text{and} \quad N_2(y) = \sum_{k=1}^{n}\sum_{\mu=1}^{\infty} y^{(k-1)}(a_\mu,\lambda) w_{i,k}^{[\mu]}.$$

We then have that our characteristic function $\triangle(\lambda)$ is

$$(36) \quad \triangle(\lambda) = |M_i(y_i)| = |N_1(y_i) + N_2(y_i)|$$

where y_i are the solutions of (1).

We shall now assume that $|N_1(y_i)|$ is sufficiently small so that it may be neglected; i.e. so that it will not affect the coefficients of the various powers of λ in $\triangle(\lambda)$ to the extent that it makes them all vanish. This can happen in the following manner.

Since $y_i^{(k-1)}$ is finite there exists some $\mu(\lambda)$ such that $|y_i^{(k-1)}(t,\lambda)| \leq \mu(\lambda)$ for t in (c,d) and λ in R'. We thus have

$$(37) \quad \left| {}_c\int^d y_i^{(k-1)}(t,\lambda)\, d\,\sigma_{i,k}(t) \right| \leq \mu(\lambda) {}_c\int^d |d\,\sigma_{i,k}(t)| \leq \mu(\lambda)\, V_c^d[\sigma_{i,k}(t)]$$

where $V_c^d[\sigma_{i,k}(t)]$ is the total variation of $\sigma_{i,k}(t)$ in (c,d) and thus

$$|N_1(y_i)| \leq n\mu(\lambda)\, V_c^d[\sigma_{i,k}(t)].$$

Now if the $V_c^d[\sigma_{i,k}(t)]$ is sufficiently small we may neglect $|N_1(y_i)|$. Under this condition we need only consider $M_i(y) = N_2(y)$.

Let $y_1(x,\lambda), \ldots, y_n(x,\lambda)$ denote a set of n distinct solutions of (1), with $f(x) = 0$; these will be the elements of the first column of the matrix $Z(x,\lambda) = {}_rY^{-1}(c,\lambda)\,{}_rY(x,\lambda)$ referred to as the solution of the homogeneous system (B) associated with the equation (1), with $f(x) = 0$.

Upon substituting these elements into $N_2(y_i)$, recalling that $z_{i,j}(c,x,\lambda) = z_i^{(j-1)}(c,x,\lambda)$, we see that our characteristic function $\triangle(\lambda) = |M_i(y_i)| = |N_2(y_i)|$ is the equation (33).

Thus the following Theorem can be stated:

THEOREM IV.

The non-homogeneous boundary value problem (1),(1a) is possible for every λ in R' providing

1) $|N_1(y_i)|$ is sufficiently small,

2) $|\lambda| \geq \rho_1$; ρ_1 a fixed number depending upon the choice of R',

3) there exists some $G_\nu \neq 0$, $[\nu < m]$, and

4) the constants $w_{i,j}^{[\mu]}$ determined by the jumps of the function $\tau_{i,k}(t)$ are of such a nature that (37) converges.

If the points of discontinuity of the function $\tau_{i,k}(t)$ are only finite in number, then we would have

$$(38) \qquad M_i(y) = \sum_{k=1}^{n} \sum_{\mu=1}^{n} y^{(k-1)}(a_\mu, \lambda) w_{i,k}^{[\mu]}$$

and upon substituting the solutions into this equation we see that our characteristic function, $\triangle(\lambda)$, is the same as the characteristic function for the n-point problem. Thus restriction (4) in the above Theorem can be omitted; i.e.

Corollary 2.

If the points of discontinuity of the function $\tau_{i,k}$ are only finite in number, the non-homogeneous boundary value problem (1),(1a) is possible for every λ in R' providing

1) $|N_1(y_i)|$ is sufficiently small,

2) $|\lambda| \geq \rho_1$, and

3) there exists some $G_\nu \neq 0$, $[\nu < m]$.

Vita

Kaj Leo Nielsen was born December 3, 1914, in Hasle, Bornholm, Denmark. He attended both public and private schools at Rønne until the age of eleven. In March, 1926 he came to the United States and lived in Detroit, Michigan for a year and a half; he attended the Detroit Public Schools and River Rouge Junior High School. He then moved to Rogers City, Michigan and was graduated from the Rogers City High School in 1932. He attended the University of Michigan from 1932 to 1936, where he received the degree of Bachelor of Arts, with High Distinction and Honors in Mathematics, and the State of Michigan High School Life Teacher's Certificate. In his Senior year he was, for five weeks, an Interne at the Grosse Pointe (Michigan) High School. He was elected to the Phi Eta Sigma, Phi Kappa Phi, and Phi Beta Kappa Fraternities. In the summer of 1936 he attended the University of Michigan Graduate School. He was a graduate student, Freshmen Adviser in the Dean of Men's Office, and Summer School Instructor at Syracuse University from 1936 to 1937, receiving the degree of Master of Arts in August, 1937 and being elected to the Pi Mu Epsilon Fraternity. From 1937 to 1940 he was an Assistant in Mathematics at the University of Illinois. He is a member of the American Mathematical Society.